

CH 1 Lines and Circles

Objectives

- Know and use the properties of a figure formed by a circle and the two tangents to this circle issued from the same point.
- Define the tangent to a circle (C) of center O at a point A as the perpendicular to OA at A.
- Recognize the position of a line with respect to a circle.
- Know the number of tangents, from a point to a circle, according to the position of this point with respect to this circle.
- Show that, if from a point outside the circle we draw two tangents to this circle, then the line joining this point to the center of the circle is an axis of symmetry of the obtained figure.

Preparatory activities

Activity 1

Required time: 5 minutes

Objectives: We expect that the students fully understand this property: that in a circle of center O, the perpendicular at a point A of the circle, to the radius OA is the tangent to this circle. Evidently, the required proof is difficult for the students, because it relies on contradiction: if B is another meeting point, OAB will be an isosceles triangle ($OA = OB = \text{radius}$), and the angle $\hat{A}$ is 90° which is impossible). But we believe the students can understand this proof.

Procedure: Group work. The teacher does the checking. She/he can suggest the study of a contrary case: What happens if there exists another meeting point B?

Activity 2

Required time: 5 minutes

Objectives: Discover the number of tangents drawn from a point on the circle or outside the circle and to perceive that from a point inside the circle no tangents can be drawn.

Procedure: Individual work by each student. The teacher makes the checking.

Activity 3

Required time: 5 minutes

Objectives: To find the relation between the distance from the center of a circle to a line and the position of this line with respect to this circle.

Procedure: Individual work by each student and the teacher makes the checking.

Activity 4

Required time: 3 minutes

Objectives: We expect the students to perceive that the figure formed by a circle of center O and the two tangents issued to this circle from point A outside the circle has OA as an axis of symmetry.

Procedure: Collective work. The teacher asks the students to read the question and chooses any one to answer. Then she writes the answer(s) on the board, and comments at the end.

Activity 5

Required time: 7 minutes

Objectives: To learn the construction of a tangent to a circle from a point on this circle or outside it.

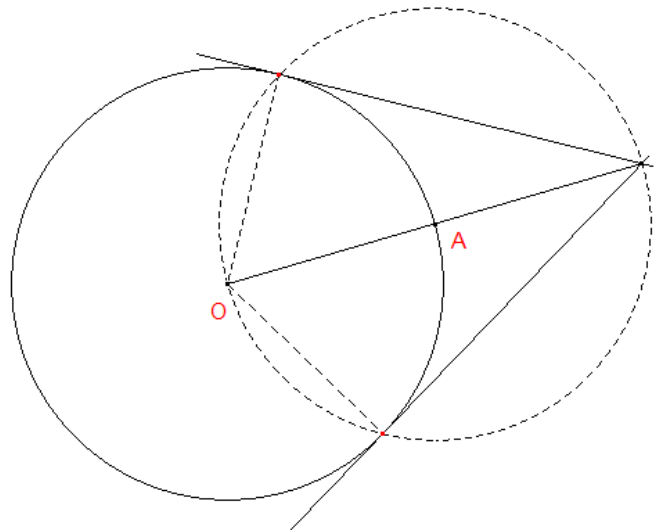
Procedure: Individual work by each student. The teacher checks the result.

At the end of these activities the teacher checks what the students have discovered (or not?). It is strongly recommended that the teacher makes a connection between the results of the students' work and the text.

Solutions for exercises

1.

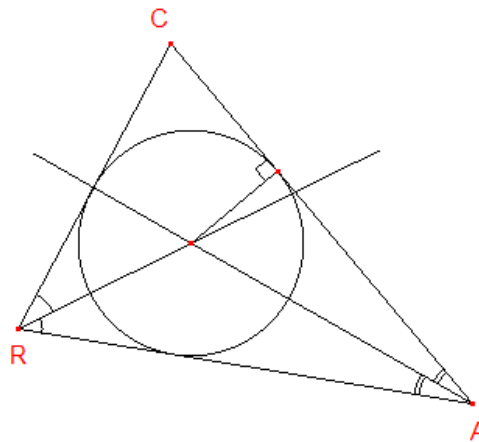
a) Figure.



b) Draw the circle of diameter $[OI]$, it cuts (C) at E and F. (IE) and (IF) are tangents to (C) .

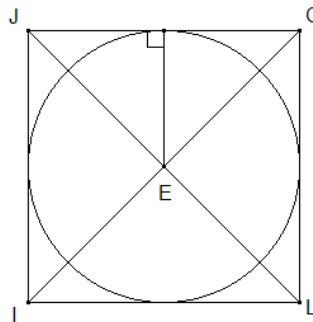
2.

a) It is the meeting point of the interior bisectors. b) Construction.



3.

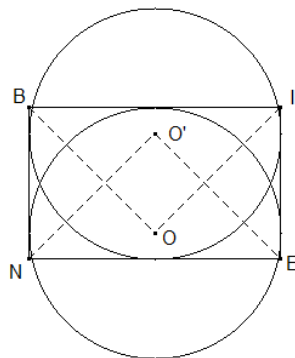
a) Construction.



b) This circle is tangent to the other sides of JOLI since the distance from E to any side is the same.

Circle Tangent Example

4.

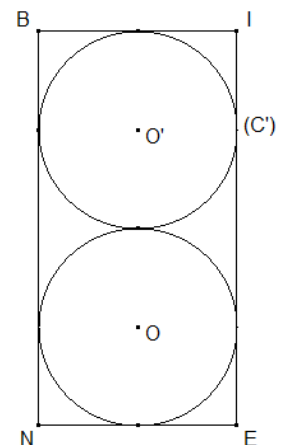


a) The center of (C) is the point of intersection of the bisectors of angles B and I, its diameter is 4 cm.

b) The center O' of (C') is the point of intersection of the bisectors of N and E, its diameter is 4 cm.

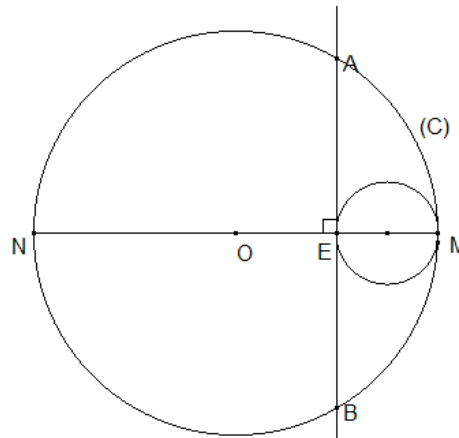
c) $IE = 8$. Since $OO' = R + R'$, the 2 circles are externally

Tangency of Two Circles



5.

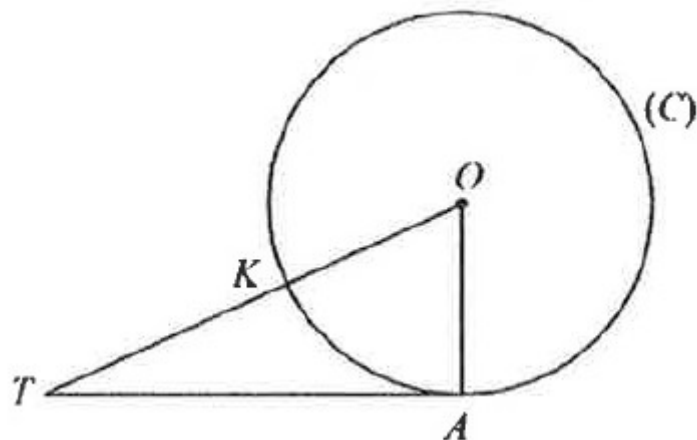
Let I be the midpoint of $[ME]$. The circle of center I and radius $IM = 1/2$ cm is tangent to (AB) and (C) .



6.

- a) (d) is exterior to (C) .
- b) (d) cuts (C) into 2 distinct points.
- c) (d) is tangent to (C) .

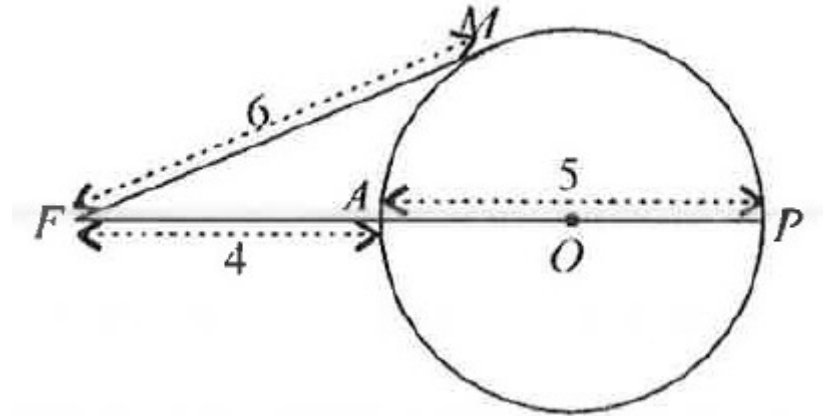
7.



- a) $TA^2 = 100 - 36 = 64$.
- b) $TO^2 = 100 + 36 = 136$.
- c) $OA = 1/2 i$, then $i = 12$.

d) $l = TK + KO = 17$; $TA^2 = (17)^2 - 64$.

8.



The radius is $5/2 = 2.5$ cm. $FO = 4 + 5/2 = 13/2$.

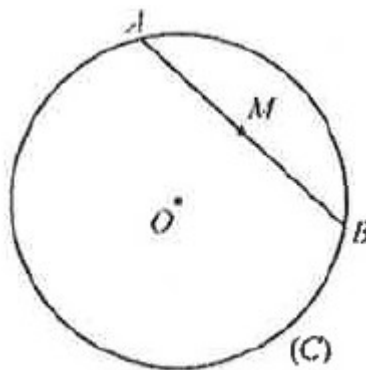
$$FO^2 = \frac{169}{4}$$

$$FM^2 + MO^2 = 36 + 25/4 = 144 + 25/4 = 169/4 = FO^2$$

Therefore, $FM \perp MO$.

Circle with radius and perpendicular lines

9.

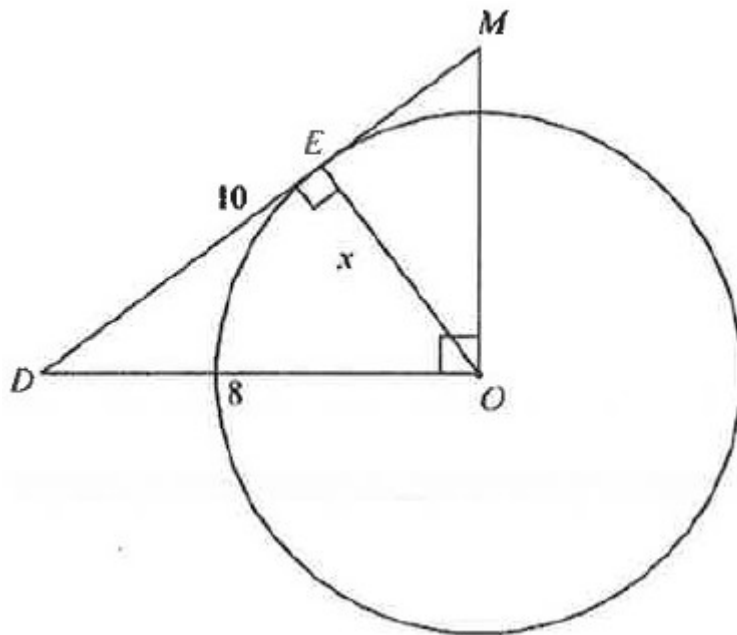


$OA = OB$; then (OM) is perpendicular to (AB) .

10.

$AB^2 = 49 - 36 = 13$; then $AB = \sqrt{13}$, the distance from A to (AB).

11.



$OM^2 = 100 - 64 = 36$; then $OM = 6$.

$$x^2 = 36 - ME^2 = 64 - DE^2$$

$$64 - DE^2 = 36 - ME^2$$

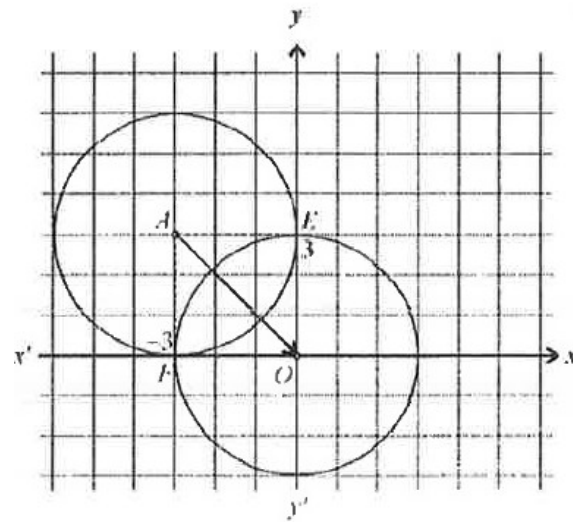
$$28 = (DE + EM)(DE - EM) = 10(DE - EM);$$

$$DE - EM = 2.8, \quad DE + EM = 10$$

Thus $DE = 6.4$ $x^2 = (64) - (6.4)^2$ so $x = 4.8$

Geometric configuration

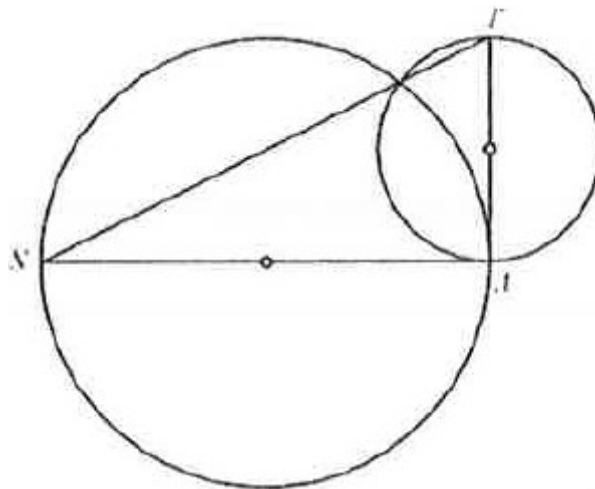
12.



- a) (C) is also tangent to $(y'y)$.
- b) O is the image of A by this translation: the circle (C') has O as center and 3 as radius. The tangents through A to this circle are (AE) and (AF) (see figure).

Circle Tangency with Translation

13.

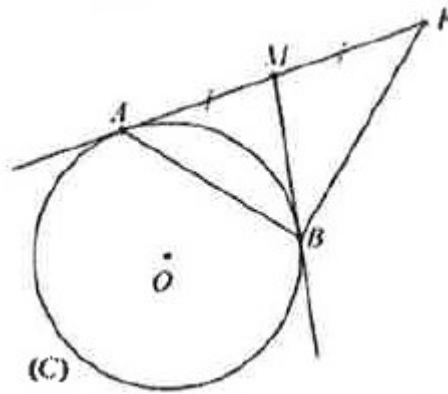


- a) $TN^2=45=36+9:$

Then $\widehat{\tan}=90^\circ$.

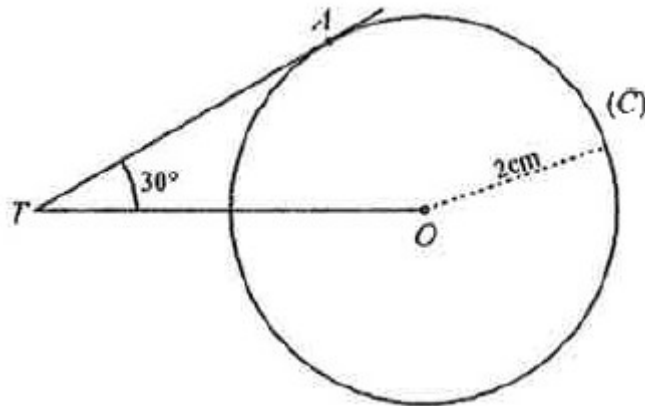
- b) (AN) is tangent to (C') and (AT) is tangent to (C'') since (TA) is perpendicular to (AN) .

14



- $MB=MA$ and $MA=MF$; then $MB=\frac{1}{2}AF$.
- So, $\widehat{MBA}=90^\circ$.

15.

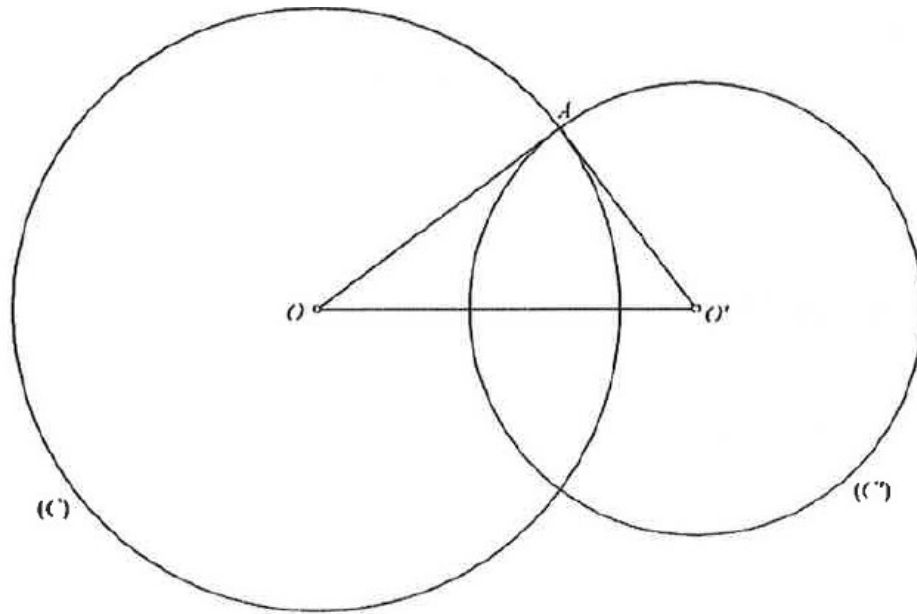


- a) The radius of the circle is 2 cm, (TA) is tangent to this circle, and $\widehat{ATO}=30^\circ$.
- b) $\widehat{ATO}$ is a semi-equilateral triangle:

$$AO = \frac{1}{2}i; \text{ then } i = 4 \text{ cm,}$$

$$TA = \frac{\sqrt{3}}{2} = 2\sqrt{3} \text{ cm.}$$

16.

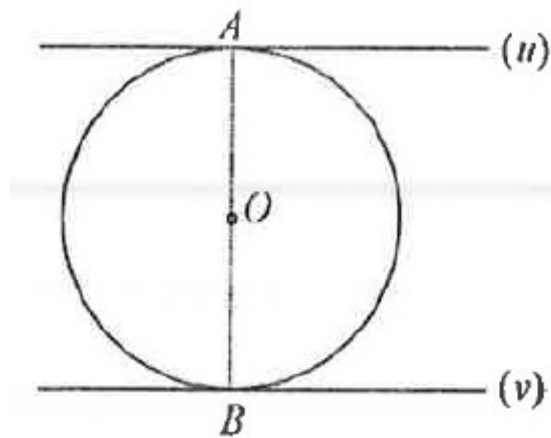


a) Construction.

b) $OO'^2 = 25 = OA^2 + O'A^2;$

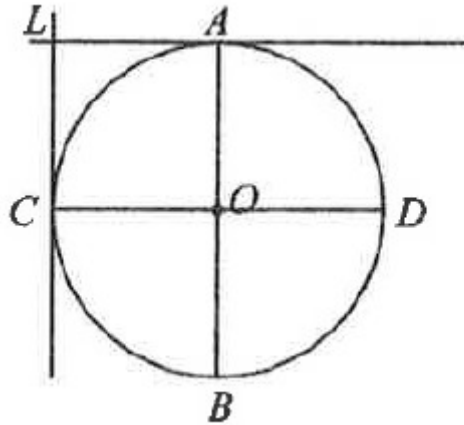
Then $\widehat{OAO'} = 90^\circ$.

17.



- $(OA) \perp (u)$.
- $(OB) \perp (v)$.
- (u) and (v) are parallel, so A , O , and B are collinear, and $[AB]$ is a diameter of (C) .

18.



- Geometric construction and drawing.
- $ALCO$ is a square.

19.

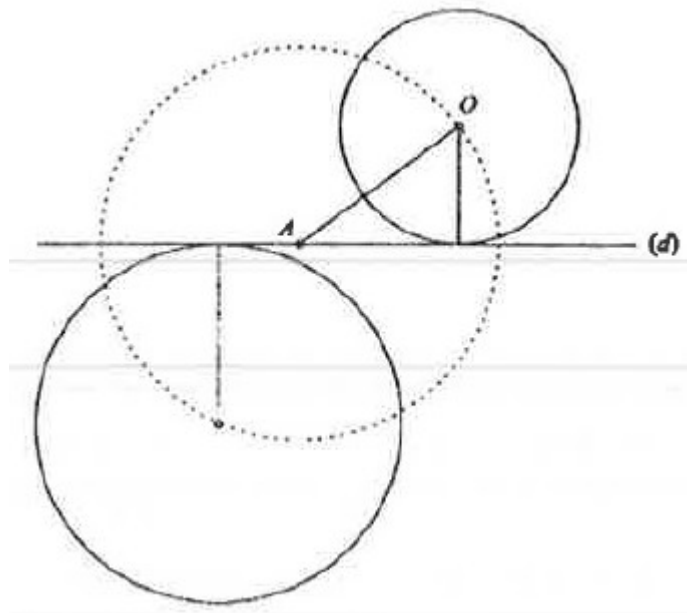
- (OS) is perpendicular to (LS) and (OS) is perpendicular to (SI) .
- The two circles are tangent to (OS) at S , and thus they are tangent to each other.

20.

$$MO - MH = MO - MI = OI = R = \text{constant}.$$

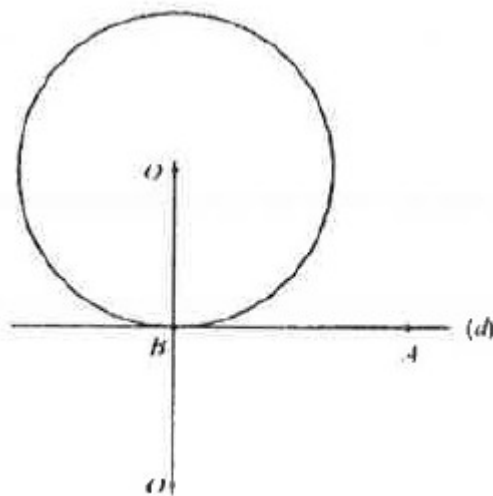
Solutions to Problems

1.



- a) Drawing.
- b) We take O on the circle of center A with radius 10 cm.
- c) The locus of O is the circle of center A with radius 10 cm.

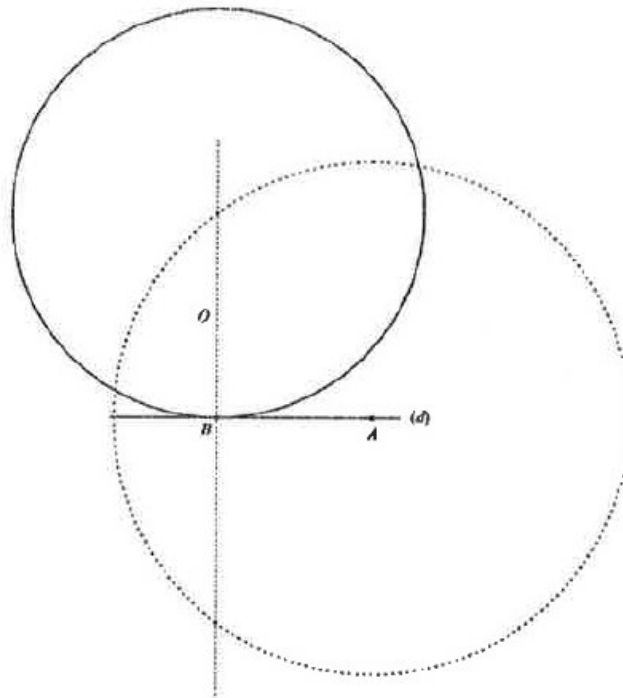
2.



- a) Drawing.
- b) On the perpendicular at B to (d) , locate the point O such that $OB=4$ cm . (2 solutions).
- c) Calculate OA^2 :

$$OA^2 = OB^2 + BA^2 = 16 + 36 = 52.$$

3.



- a) Drawing.
- b) O is the intersection of the perpendicular through B to (d) and the circle of center A with radius 10 cm. (2 solutions).
- c) Calculate OB^2 :

$$OB^2 = OA^2 - AB^2 = 100 - 36 = 64, OB = 8 \text{ cm}.$$

- d) Case analysis:

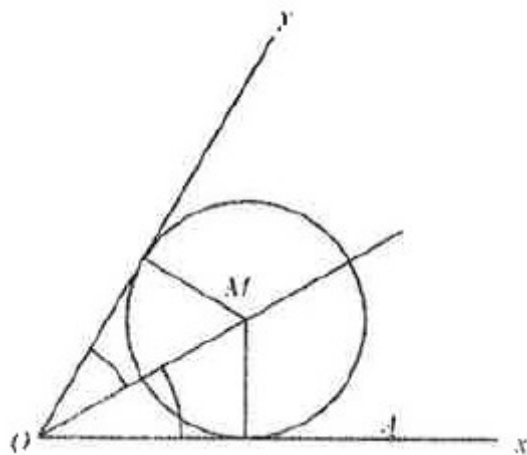
- For $R=8$ cm, we have the same case as (b).
- For $R=5$ cm, we have:

$$100 \neq 25 + 36 \text{ (no solutions).}$$

4.

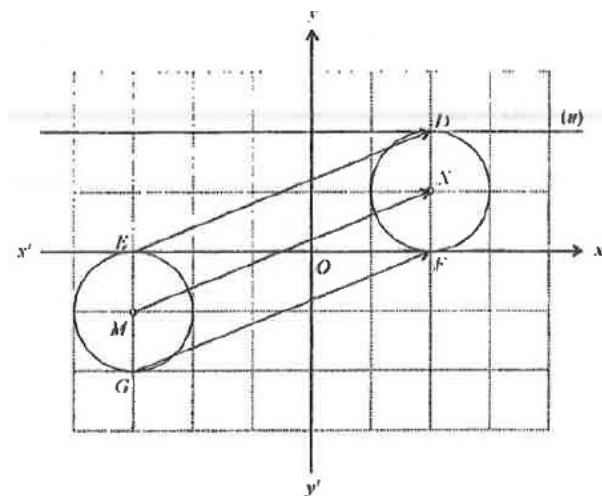
- O belongs to one of the lines parallel to (AB) and at a distance of 6 cm from (AB) .
- O belongs to the perpendicular through B to (d) .

5.



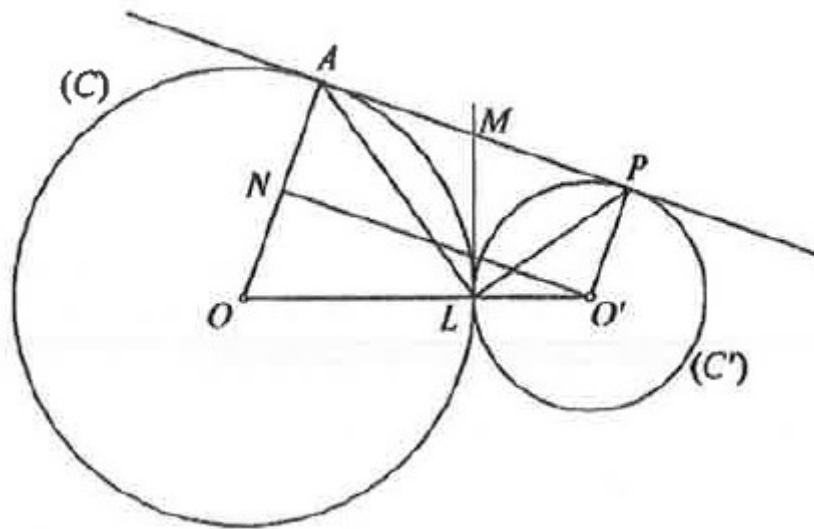
The locus of M is one of the bisectors of angle $x\hat{O}y$. The locus of N is a line passing through the midpoint of $[OA]$ and parallel to the locus of M .

6.



- Construction.
- Construction.

- ## 7.



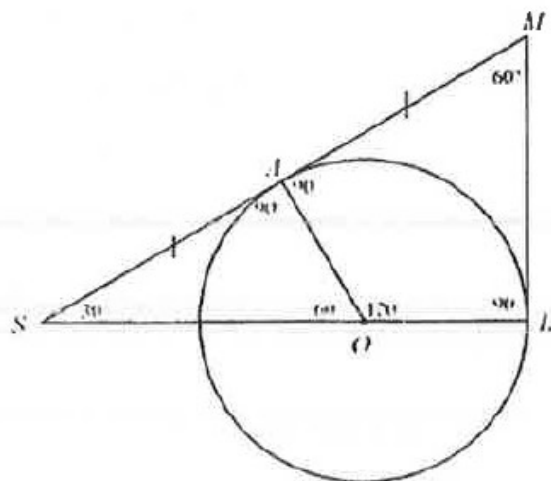
- $$O'N^2 = OO^2 - ON^2 = 36 - 4 = 32; O'N = 4\sqrt{2}.$$

$$ML = \frac{PA}{2} = 2\sqrt{2}.$$

- d) E is the midpoint of $[OO']$.

$$ME = \frac{4+2}{2} = 3 = \frac{1}{2} OO'.$$

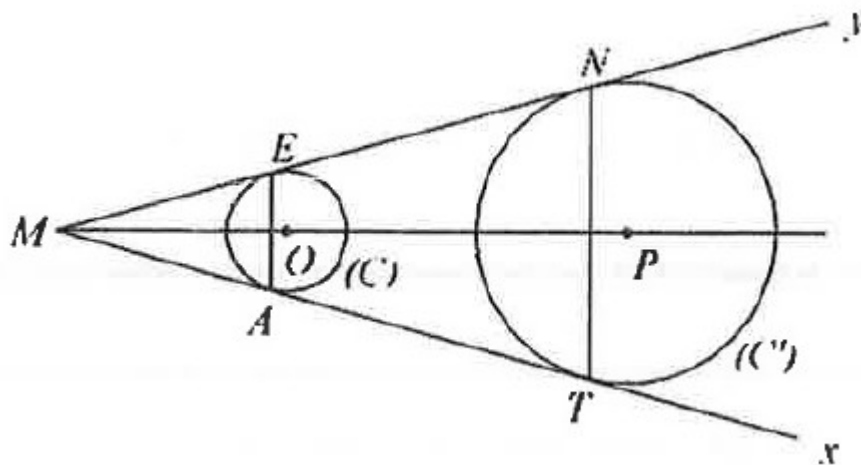
8.



a) $ML = \frac{1}{2}MS$, then $\hat{S} = 30^\circ$ and $\hat{M} = 60^\circ$.

b) $\widehat{ACS} = 60^\circ$.

9.

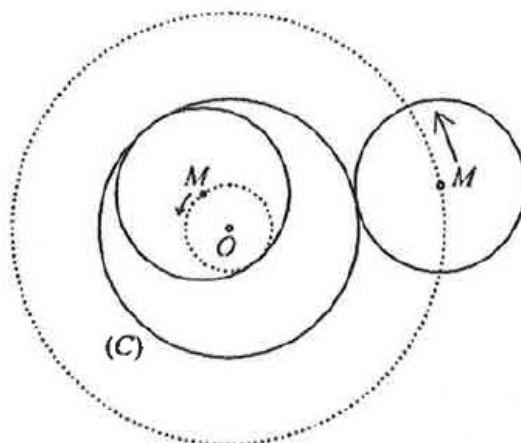


a) Points M , O , and P belong to the bisector of $\widehat{xOy}$, hence they are collinear.

b) $AT = MT - MA = MN - ME = EN$.

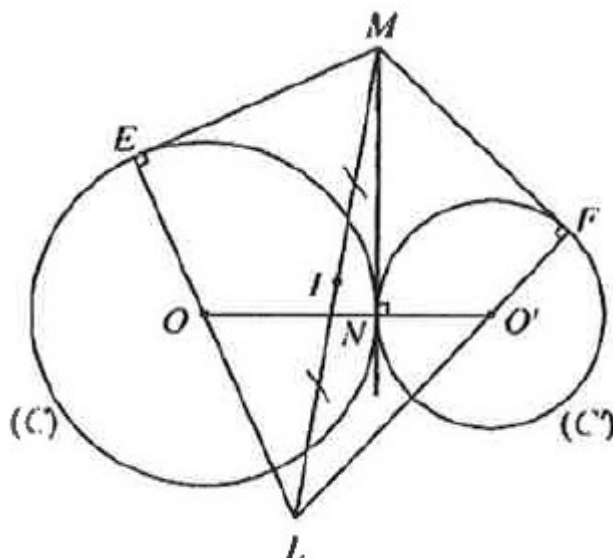
c) The triangles MEA and MNT are isosceles with a common angle M , hence $(EA) \parallel (NT)$. Thus, $ENTA$ is an isosceles trapezoid.

10.



- M varies on the circle with center O and radius 5 cm if the circle (M) is externally tangent to (C) .
- M moves on the circle with center O and radius 1 cm if the circle (M) is internally tangent to (C) .

11.

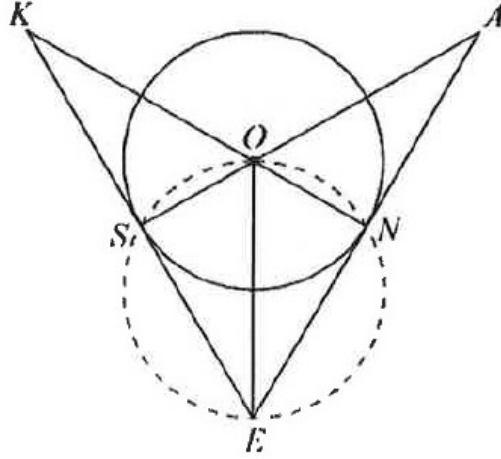


- a) (C) and (C') are tangent at N . Lines (MN) , (ME) , and (MF) are tangent.
 $\widehat{OEM} = \widehat{O'FM} = 90^\circ$.

L is the intersection of (OE) and $(O'F)$, and I is the midpoint of $[ML]$.

- b) $ME=MN$ and $MN=MF$.
- c) $IE=IF=\frac{ML}{2}$, and $\widehat{MEL}=\widehat{MFL}=90^\circ$.
- d) $LE^2=ML^2-ME^2=ML^2-MF^2=LF^2$.

12.



- a) $\widehat{OSE}=\widehat{ONE}=90^\circ$, and $OS=ON=\frac{1}{2}OE$. Therefore:
 $\widehat{SON}=\widehat{EON}=60^\circ$ and $\widehat{SON}=120^\circ$.
- b) (EN) is perpendicular to (ON) , and (ES) is perpendicular to (OS) .

$$ES=EN=\frac{4\sqrt{3}}{2}=2\sqrt{3}.$$

- c) The area of $OSEN$ is:

$$\text{Area of } OSEN=2 \times \text{Area of } OSE=OS \times SE=4\sqrt{3}.$$

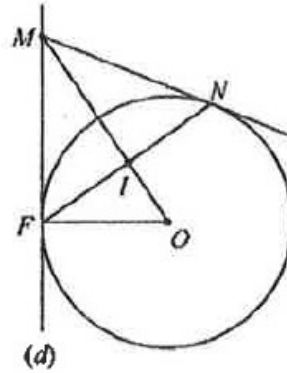
- d) $ES=EN$, E is a common angle, and $\hat{S}=\hat{N}=90^\circ$.

Then, the two triangles ESA and ENK are congruent, hence:

$$AS=KN, EK=EA, \hat{K}=\hat{A}.$$

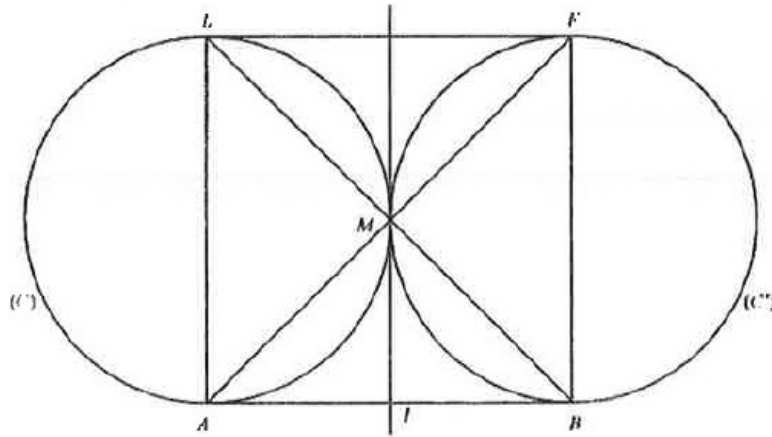
- e) Since $EK=EA$ and $\hat{E}=60^\circ$, EAK is equilateral.
- f) O is the center of triangle EAK , and (C) is the inscribed circle of this triangle, so (KA) is tangent to (C) .

13.



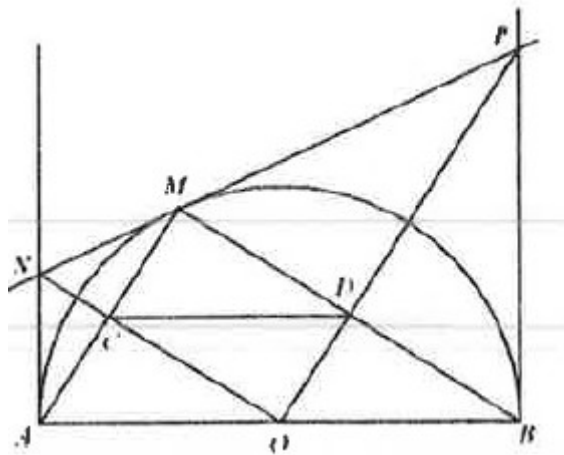
- a) $OF=ON$ and $MF=MN$. Therefore, (MO) is the perpendicular bisector of $[FN]$.
- b) $\widehat{OIF}=90^\circ$. Thus, the locus of I is the circle with diameter $[OF]$, and the locus of E is the circle with center O and radius 1.5 cm.

14.



- a) $R=3\text{cm}$.
- b) $IA=IB$. I is the midpoint of $[AB]$, and MAB is a right isosceles triangle.
- c) $\widehat{A}=90^\circ$. Hence, $[BF]$ and $[AL]$ are diameters in (C'') and (C') , respectively. $BALF$ is a square.

15.



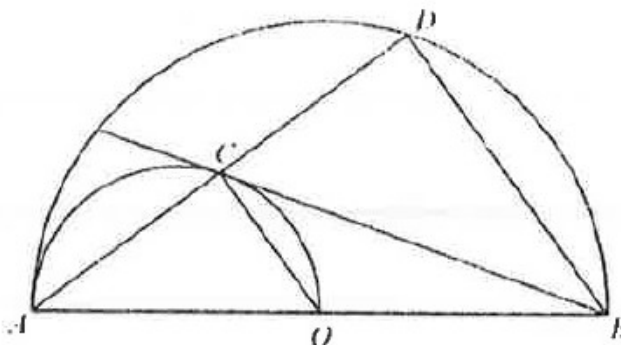
- a) Figure.

$$\widehat{COD}=90^{\circ}.$$

- b) $MCOD$ is a rectangle.

$$CD=OM=R.$$

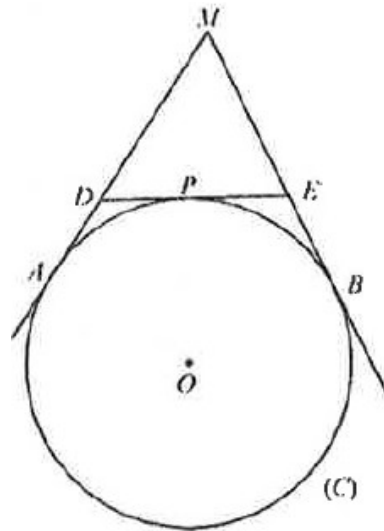
16



- a) $\widehat{AC'O} = \widehat{ADB} = 90^\circ$, then (OX') is parallel to (BD) , but O is the midpoint of $[AB]$; hence C' is the midpoint of $[AD]$.

- b) l is the midpoint of $[AO]$ and (IC') is perpendicular to (BC') . (DO) is parallel to (IC') ; therefore, (DO) is perpendicular to (BC') .

17.

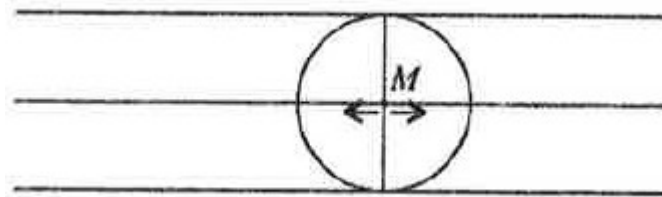


a) The midpoint of $[PB]$ moves on an arc of the circle of diameter $[OB]$.

b) Perimeter of triangle MDE :

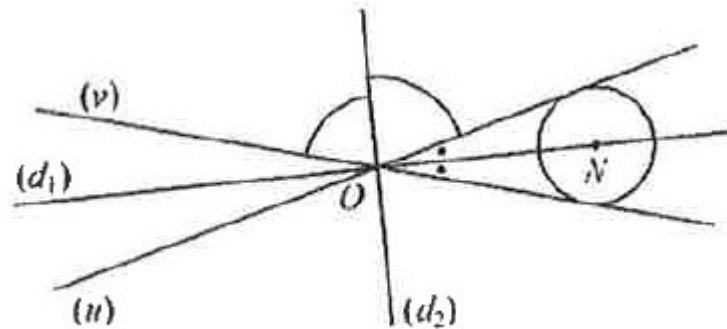
$$MDE = MD + DP + PE + ME = MD + DA + EB + ME = MA + MB = 2 MA.$$

18.



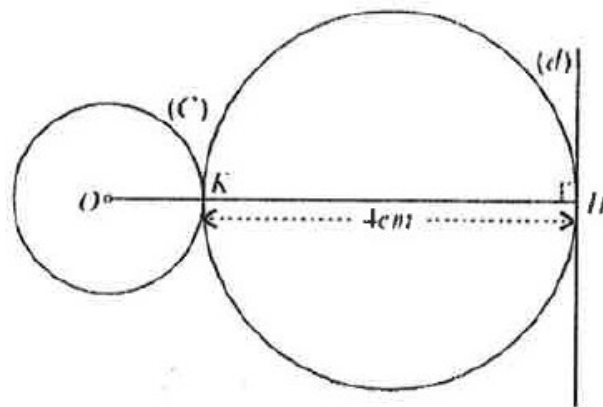
The locus of M is a line parallel to the two given lines and located halfway between them.

19.



The locus of the midpoint of $[ON]$ is the interior or exterior bisector of the angle formed by the two given lines.

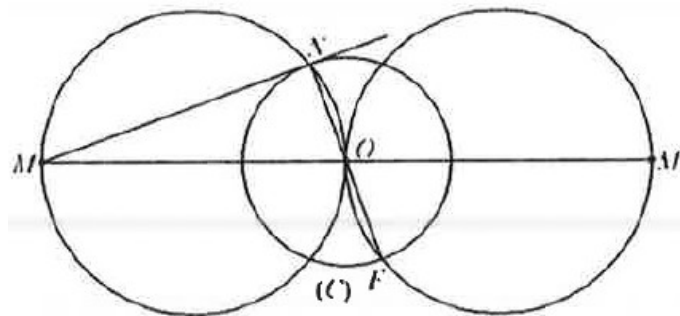
20.



The center O' of the circle (C') belongs to a line $d' \parallel d$, such that the distance between d and d' is 2 cm.

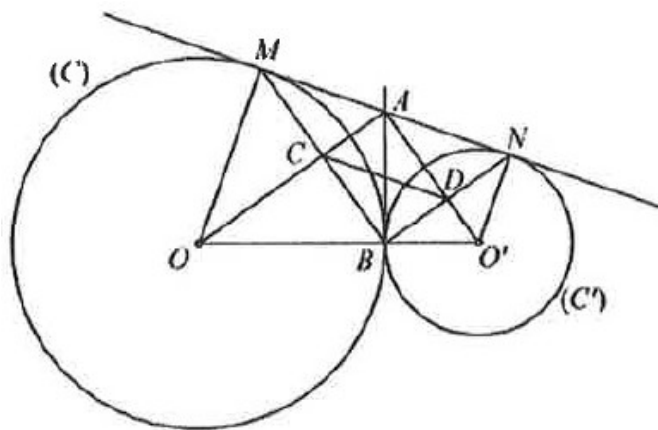
O' also belongs to the circle with center O and radius 3 cm. N moves on the line passing through the midpoint of $[OH]$ and parallel to (d) .

21.



- a) The locus of N is the circle of diameter $[MO]$.
- b) The locus of F is the circle of diameter $[M'O]$, where M' is the symmetric of M with respect to O .

22.



- a) Figure.
- b) (OA) and $(O'A)$ are the perpendicular bisectors of $[MB]$ and $[NB]$.
 C is the midpoint of $[MB]$ and D is the midpoint of $[BN]$, thus in triangle MBN , we have $(MN) \parallel (CD)$ and $MN = 2 CD$.